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A generalization of Hagopian's theorem and exponents ☆

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Abstract

We generalize Hagopian's theorem characterizing solenoids to higher dimensions by showing that any homogeneous continuum admitting a fiber bundle projection onto a torus with totally disconnected fibers admits a compatible topological group structure. And then the higher dimensional exponent group is introduced. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A space X is a *continuum* if it is compact, connected and metrizable, and X is *homogeneous* if given any $x, y \in X$ there is a homeomorphism $h : X \to X$ with $h(x) = y$. We show that if X is a homogeneous continuum which (for some countable $\kappa \geqslant 1$) admits a fiber bundle projection $p : X \to \mathbb{T}^\kappa = (\mathbb{R}/\mathbb{Z})^\kappa$ with totally disconnected fibers, then X also admits a compatible Abelian topological group structure. This generalizes a weakened version of the following theorem of Hagopian: If every subcontinuum of the homogeneous continuum X is an arc, then X is homeomorphic to a solenoid (which includes the circle as a possibility) [8]. For $\kappa = 1$ the theorem currently under consideration follows from Hagopian's theorem and applies to all compact one-dimensional minimal sets of flows (see [2]). In higher dimensions, spaces admitting such a fiber bundle structure arise naturally in two settings: as minimal sets of foliations and as limit sets of discrete dynamical systems (see [15]).

A good example of an S^1 fiber bundle which is not homogeneous is the minimal set of a Denjoy flow on a torus: the path components are not "evenly spaced", and so there is no

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Effros homeomorphism for sufficiently small numbers. If one follows path components far enough along in one direction, eventually they will spread apart. Our result shows that this situation persists in higher dimensions.

We then generalize the exponent group introduced in [5] to higher dimensions. In what follows we use the terminology and results of [13], and $\pi^\kappa : \mathbb{R}^\kappa \to \mathbb{T}^\kappa$ denotes the standard fibration with unique path lifting $\langle t_i \rangle_{i=1}^{\kappa} \mapsto \langle t_i \bmod 1 \rangle_{i=1}^{\kappa}$ and d is a metric for X.

Note. After the original submission of this paper, Sadun and Williams [12] have shown that finite-dimensional tiling spaces admit a fiber bundle projection with totally disconnected fibers onto a torus of the same dimension as the tiling.

2. Homogeneity

Theorem 2.1. *If X is a homogeneous continuum and if X admits a fiber bundle projection $p : X \to \mathbb{T}^\kappa$ with totally disconnected fibers, then X admits a compatible Abelian topological group structure.*

Proof. Suppose $p : X \to \mathbb{T}^\kappa$ is a fiber bundle projection with totally disconnected fiber F for some $\kappa > 1$. Then p is a fibration with unique path lifting (see [13, 2.2.5, 2.7.14]), and so the natural group action

$$\alpha : \mathbb{R}^\kappa \times \mathbb{T}^\kappa \to \mathbb{T}^\kappa$$

given by

$$\alpha(s, x) = \pi^\kappa(s) + x$$

lifts uniquely to an action $\widetilde{\alpha}$ on X

$$\begin{tikzcd}
\mathbb{R}^\kappa \times X \arrow[r, "\widetilde{\alpha}"] \arrow[d, "\mathrm{id} \times p"'] & X \arrow[d, "p"] \\
\mathbb{R}^\kappa \times \mathbb{T}^\kappa \arrow[r, "\alpha"] & \mathbb{T}^\kappa
\end{tikzcd}$$

For, given any $(s, x) \in \mathbb{R}^\kappa \times X$, we have the interval $I = [t \cdot s, (1 - t) \cdot s]$ $(t \in [0, 1])$ and the following commutative diagram

$$\begin{tikzcd}
\{\mathbf{0}\} \times X \arrow[rr, "\text{“id”}"] \arrow[d, "\cap"] & & X \arrow[d, "p"] \\
I \times X \arrow[r, "\mathrm{id} \times p"] & I \times \mathbb{T}^\kappa \arrow[r, "\alpha"] & \mathbb{T}^\kappa
\end{tikzcd}$$

and so the fibration property allows us to define $\widetilde{\alpha} \upharpoonright I \times X$ as needed. To see that $\widetilde{\alpha}$ is continuous, we can replace I in the diagram with increasing open connected subsets of $\mathbb{R}^\kappa$ whose union is all of $\mathbb{R}^\kappa$.

Then for a given point $e \in X$,

$$p(e) + \pi^\kappa = p \circ \widetilde{\alpha} \upharpoonright \mathbb{R}^\kappa \times \{e\}$$

is a fibration and p has unique path lifting, and so it follows from Lemma 2.1 of [4] that $\widetilde{\alpha} \upharpoonright \mathbb{R}^\kappa \times \{e\}$ is a fibration. Hence, the $\widetilde{\alpha}$-trajectory of e is the path component of e in X, [13, 2.3.1]. We now proceed to show that the path components of X are dense in X.

The proof is similar to the proof of a related fact for matchbox manifolds given in [1]. Each point of X is contained in an open set homeomorphic to $V \times F$ for some open subset V of $\mathbb{T}^\kappa$. Since F is compact and totally disconnected, there is a basis $\mathcal{B}$ for X consisting of sets homeomorphic to $U \times Z$ with U open and connected in $\mathbb{T}^\kappa$ and Z a closed and open subset of F. Let $B \approx U \times Z$ be any element of $\mathcal{B}$. We wish to show that $PC(B)$ (the union of all the path components of X which meet B) is both open and closed in X. For $t \in \mathbb{R}^\kappa - \{\mathbf{0}\}$, $\widetilde{\alpha}$ induces the non-singular flow

$$\widetilde{\alpha}[t]:\mathbb{R} \times X \to X, \quad \widetilde{\alpha}[t](r,x) = \widetilde{\alpha}(tr,x) \qquad \text{(and similarly for } \alpha).$$

For $x \in PC(B)$ we may then choose $s \in \mathbb{R}^\kappa$ so that $\widetilde{\alpha}(s,x) = y \in B$ and so that the corresponding linear flow $\alpha[s]$ (and hence $\widetilde{\alpha}[s]$) is aperiodic, which is possible since $\kappa > 1$ and since the collection of $t \in \mathbb{R}^\kappa - \{\mathbf{0}\}$ for which $\alpha[t]$ is aperiodic is dense. By a theorem of Bebutov (see [10, V.2.15]) we may construct a flow box joining y and x which contains both points in its interior. Hence, all points in a neighborhood of x are also in $PC(B)$, demonstrating that it is open.

Suppose then that $x \in \overline{PC(B)}$ with $(x_n)_n \to x$ and $\{x_n\} \subset PC(B)$. The path component of x_n meets B in a point $b_n \approx (u_n, z_n)$ and since U is path connected, we may assume that $u_n = u$ for some $u \in U$ and for all n. The compactness of Z allows us to find a subsequence of (z_n) converging to some $z \in Z$ with $b \approx (u,z) \in B$. Then there is some $\varepsilon > 0$ so that $B_d(b,\varepsilon) \subset B$. Applying the Effros theorem [6,14,7] to $H(X)$, the group of homeomorphisms of X in the sup metric, there is a $\delta > 0$ so that for any $v,\ w \in X$ with $d(v,w) < \delta$ there is an $h^{[v,w]} \in H(X)$ with $h^{[v,w]}(v) = w$ and $d(p, h^{[v,w]}(p)) < \frac{1}{2}\varepsilon$ for all $p \in X$ (denoted: $\delta \overset{\text{Eff}}{\sim} \frac{1}{2}\varepsilon$). We now choose n so that $d(b_n, b) < \frac{1}{2}\varepsilon$ and $d(x_n, x) < \delta$ and a corresponding homeomorphism $h^{[x_n,x]}$. Then $h^{[x_n,x]}(b_n) \in B_d(b,\varepsilon) \subset B$ is a point in the path component of x, demonstrating that $x \in PC(B)$, and so $PC(B) = X$. Since B was any such basis element, each path component of X is dense.

With $\widetilde{\alpha}_x : \mathbb{R}^\kappa \to X$ denoting the $\widetilde{\alpha}$-orbit of $x \in X$ and giving

$$A \overset{\text{def}}{=} \left\{\widetilde{\alpha}_x \mid x \in X\right\}$$

the topology it inherits from the collection of all maps $\mathbb{R}^\kappa \to X$ in the sup metric, we proceed to show that $h : X \to A;\ h(x) \overset{\text{def}}{=} \widetilde{\alpha}_x$ is a homeomorphism. Since h is clearly a bijection and X is compact, it suffices to show that h is continuous. Given $(x_n)_n \to x$ in X and $\varepsilon > 0$ we need to find an N so that $\sup_{s \in \mathbb{R}^\kappa} d(\widetilde{\alpha}_{x_n}(s), \widetilde{\alpha}_x(s)) < \varepsilon$ for all $n \geqslant N$.

Let d_κ be a translation invariant metric for $\mathbb{T}^\kappa$. First we find a Lebesgue number $\lambda > 0$ for a covering $\mathcal{O}$ of $\mathbb{T}^\kappa$ by open sets V satisfying

$$p^{-1}(V) \approx V \times F$$

and $p(y) = v$ for $y \approx (v, f)$. Since X and $\mathbb{T}^\kappa$ are compact, we may find a connected neighborhood U of $\mathbf{0} \in \mathbb{R}^\kappa$ so that

$$\sup_{y \in \mathbb{T}^\kappa} \bigl\{\operatorname{diam}\bigl(\alpha\bigl(U \times \{y\}\bigr)\bigr)\bigr\} < \lambda \quad \text{and} \quad \sup_{y \in X} \bigl\{\operatorname{diam}\bigl(\widetilde{\alpha}\bigl(U \times \{y\}\bigr)\bigr)\bigr\} < \tfrac{1}{3}\varepsilon$$

and so that $\alpha(U \times \{e\})$ is the η-neighborhood of e in $\mathbb{T}^\kappa$ for some $\eta > 0$. The translation invariance of d_κ yields that $\alpha(U \times \{y\})$ is the η-neighborhood of $y \in \mathbb{T}^\kappa$. Since p is uniformly continuous, there is a $\tau > 0$ so that

$$d(y, y') < \tau \Rightarrow d_\kappa\bigl(p(y), p(y')\bigr) < \eta.$$

Next we find

$$0 < \delta \overset{\text{Eff}}{\sim} \min\bigl\{\tfrac{1}{2}\varepsilon, \tau\bigr\} \text{ and } N \text{ so that } \{x_n\}_{n \geqslant N} \subset B_d(x, \delta).$$

Given $n \geqslant N$ there is an $h^{[x_n,x]} \in H(X)$ within $\min\{\frac{1}{2}\varepsilon, \tau\}$ of id_X. And so

$$p(x) = p\bigl(h^{[x_n,x]}(x_n)\bigr) \in \alpha\bigl(U \times \bigl\{p(x_n)\bigr\}\bigr)$$

since $d(x_n, x) < \tau$. For $s \in \mathbb{R}^\kappa$, the translation invariance of d_κ yields that $d_\kappa(\alpha(s, p(x)), \alpha(s, p(x_n))) < \eta$, while the choice of $h^{[x_n,x]}$ and the equality $\alpha(s, p(x_n)) = p \circ \widetilde{\alpha}(s, x_n)$ yields that

$$d_\kappa\bigl(\alpha\bigl(s, p(x_n)\bigr), p \circ h^{[x_n,x]} \circ \widetilde{\alpha}\bigl(s, (x_n)\bigr)\bigr) < \eta.$$

Combining this with the equality $\alpha(s, p(x)) = p \circ \widetilde{\alpha}(s, x)$, we obtain

$$\bigl\{p \circ \widetilde{\alpha}(s, x), p \circ h^{[x_n,x]} \circ \widetilde{\alpha}\bigl(s, (x_n)\bigr)\bigr\} \subset \alpha\bigl(U \times \bigl\{\alpha\bigl(s, p(x_n)\bigr)\bigr\}\bigr) \quad \text{for all } s \in \mathbb{R}^\kappa. \quad (*)$$

For $s \in \mathbb{R}^\kappa$ let

$$U(s) \overset{\text{def}}{=} p^{-1}\bigl(\alpha\bigl(U \times \bigl\{\alpha\bigl(s, p(x_n)\bigr)\bigr\}\bigr)\bigr).$$

By its size, we know that $\alpha(U \times \{\alpha(s, p(x_n))\})$ fits inside some $V \in \mathcal{O}$, and so

$$U(s) \approx \alpha\bigl(U \times \bigl\{\alpha\bigl(s, p(x_n)\bigr)\bigr\}\bigr) \times F$$

and for $y \approx (v, f)$, $p(y) = v$. Then by $(*)$ we have

$$\bigl\{\widetilde{\alpha}(s, x), h^{[x_n,x]} \circ \widetilde{\alpha}\bigl(s, (x_n)\bigr)\bigr\} \subset U(s).$$

Let

$$W \overset{\text{def}}{=} \bigl\{s \in \mathbb{R}^\kappa \mid \widetilde{\alpha}\bigl(s, (x)\bigr) \text{ and } h^{[x_n,x]} \circ \widetilde{\alpha}\bigl(s, (x_n)\bigr) \\ \text{are in the same component of } U(s)\bigr\}.$$

By construction, both W and $\mathbb{R}^\kappa - W$ are open in $\mathbb{R}^\kappa$ since the components of $U(s)$ are homeomorphic to U. Since $h^{[x_n,x]}(x_n) = x$, $\mathbf{0} \in W$ and since $\mathbb{R}^\kappa$ is connected, $W = \mathbb{R}^\kappa$. Thus, with $C(s)$ denoting the component of $U(s)$ containing both $\widetilde{\alpha}(s, (x))$ and $h^{[x_n,x]} \circ \widetilde{\alpha}(s, (x_n))$, and with

$$f_s \overset{\text{def}}{=} C(s) \cap p^{-1}\bigl(\alpha\bigl(s, p(x_n)\bigr)\bigr)$$

we have that

$$\{\widetilde{\alpha}(s,x), h^{[x_n,x]} \circ \widetilde{\alpha}(s,(x_n))\} \subset \widetilde{\alpha}(U \times \{f_s\}),$$

implying that $d(\widetilde{\alpha}(s,x), h^{[x_n,x]} \circ \widetilde{\alpha}(s,(x_n))) < \frac{1}{3}\varepsilon$ by our initial choice of U. The choice of $h^{[x_n,x]}$ yields that $d(\widetilde{\alpha}(s,x), \widetilde{\alpha}(s,(x_n))) < \frac{5}{6}\varepsilon$. Since s was any point of $\mathbb{R}^\kappa$, we may finally conclude that

$$\sup_{s\in\mathbb{R}^\kappa} d(\widetilde{\alpha}_{x_n}(s), \widetilde{\alpha}_x(s)) < \varepsilon \quad \text{for all } n \geqslant N.$$

And so h is a homeomorphism. It follows easily that $\widetilde{\alpha}$ is uniformly Lyapunov stable in the strongest sense: for any given $\varepsilon > 0$ there is a $\delta > 0$ so that

$$d(x,y) < \delta \Rightarrow d(\widetilde{\alpha}(s,x), \widetilde{\alpha}(s,y)) < \varepsilon \quad \text{for all } s \in \mathbb{R}^\kappa.$$

Fixing a point $e \in X$, the density of the orbit of e means that any $a, b \in X$ may be represented in the form $a = \lim_i \{\widetilde{\alpha}(t_i^a, e)\}$ and $b = \lim_i \{\widetilde{\alpha}(t_i^b, e)\}$. The operations

$$-a = \lim_i \{\widetilde{\alpha}(-t_i^b, e)\} \quad \text{and} \quad a + b = \lim_i \{\widetilde{\alpha}(t_i^a + t_i^b, e)\}$$

then give X a well-defined, Abelian topological group structure compatible with the original topology of X. The proof is essentially identical with that found in [10, V.8.16] and is therefore omitted. $\square$

It follows that such an X is homeomorphic to either $\mathbb{T}^\kappa$ or the inverse limit of an inverse sequence of $\mathbb{T}^\kappa$ with epimorphic bonding maps (see [4]). And it follows directly that any such X is bihomogeneous: the homeomorphism

$$x \mapsto (a+b) - x$$

switches a and b.

3. Exponents

We now define the exponent group and explore its topological significance. In what follows, $\mathrm{Hom}(\mathbb{R}^\kappa, \mathbb{R}^\kappa)$ denotes the group of continuous homomorphisms $\mathbb{R}^\kappa \to \mathbb{R}^\kappa$ with point-wise addition and $[X; Y]$ denotes the group of homotopy classes of maps $X \to Y$, and $[f]$ denotes the homotopy class of $f : X \to Y$.

Definition 3.1. Given a map $f : W \to X$, $\{w_i\}_{i=1}^\infty \subset W$ is an f-sequence if $\{f(w_i)\}_{i=1}^\infty$ converges in X.

Definition 3.2. Given a map f of $\mathbb{R}^\kappa$ into a metric space X, the exponent group of f, denoted $\mathcal{E}_f$, is

$$\{A \in \mathrm{Hom}(\mathbb{R}^\kappa, \mathbb{R}^\kappa) \mid \{\pi^\kappa(A(t_i))\}_{i=1}^\infty \text{ converges in } \mathbb{T}^\kappa \text{ for all } f\text{-sequences } \{t_i\}_{i=1}^\infty\}.$$

Lemma 3.3. *$\mathcal{E}_f$ is a subgroup of* $\mathrm{Hom}(\mathbb{R}^\kappa, \mathbb{R}^\kappa)$.

Proof. The zero homomorphism is trivially in $\mathcal{E}_f$. If A and B are in $\mathcal{E}_f$ and if $\{t_i\}_{i=1}^\infty$ is an f-sequence, then $\{\pi^\kappa(A(t_i))\}_{i=1}^\infty$ and $\{\pi^\kappa(B(t_i))\}_{i=1}^\infty$ converge in $\mathbb{T}^\kappa$. And since "$-$" is continuous on $\mathrm{Hom}(\mathbb{R}^\kappa, \mathbb{R}^\kappa) \times \mathrm{Hom}(\mathbb{R}^\kappa, \mathbb{R}^\kappa)$, this implies that $\{\pi^\kappa((A - B)(t_i))\}_{i=1}^\infty$ converges in $\mathbb{T}^\kappa$. $\square$

Theorem 3.4. *Given a map f of $\mathbb{R}^\kappa$ into a metric space X, with*

$$\iota : \mathcal{E}_f \to \left[\overline{f(\mathbb{R}^\kappa)}; \mathbb{T}^\kappa\right]$$

given by

$$A \overset{\iota}{\mapsto} [f_A], \quad \textit{where } f_A : \overline{f(\mathbb{R}^\kappa)} \to \mathbb{T}^\kappa \textit{ sends } \lim_i \bigl\{f(t_i)\bigr\} \textit{ to } \lim_i \bigl\{\pi^\kappa\bigl(A(t_i)\bigr)\bigr\}$$

we have:

(1) *ι is a homomorphism (we give $[\overline{f(\mathbb{R}^\kappa)}; \mathbb{T}^\kappa]$ the group operation induced by pointwise addition of maps).*

(2) *If $\iota(A) = \iota(B)$ and if $\mathcal{C}$ is any contractible topological subspace of $\mathbb{R}^\kappa$ satisfying the condition that the map $(A - B)_{\mathcal{C}} : \mathcal{C} \to (A - B)(\mathcal{C})$; $t \mapsto (A - B)(t)$ has a continuous inverse, then for any f-sequence $\{t_i\}_{i=1}^\infty \subset \mathcal{C}$, the sequence $\{(A - B)(t_i)\}_{i=1}^\infty$ converges in $\mathbb{R}^\kappa$.*

(3) *If $\overline{f(\mathbb{R}^\kappa)}$ is compact, then ι is an embedding and $\mathcal{E}_f$ is countable.*

(4) *If $\iota(A) = \iota(B)$ and if $\kappa = n < \infty$ and if there is an unbounded f-sequence $\{t_i\}_{i=1}^\infty$, then $A - B$ is not invertible.*

Proof. (1) Let $A \in \mathcal{E}_f$ be given. If $\lim_i\{f(s_i)\} = \lim_i\{f(t_i)\} = x$ are two representations of a point in $\overline{f(\mathbb{R}^\kappa)}$, then $\lim_i\{\pi^\kappa(A(s_i))\} = \sigma$ and $\lim_i\{\pi^\kappa(A(t_i))\} = \tau$ both exist by the definition of $\mathcal{E}_f$. Then for $i \in \{1, 2, \ldots\}$ if we define $u_{2i-1} = s_i$ and $u_{2i} = t_i$, we have that $\lim_i\{f(u_i)\} = x$, implying that $\lim_i\{\pi^\kappa(A(u_i))\}$ exists, which is only possible if $\sigma = \tau$. Thus, f_A is a well-defined function. To see that f_A is continuous, consider a convergent sequence $\{\lim_i\{f(t_i^j)\}\}_{j=1}^\infty = \{x^j\}_{j=1}^\infty \to x$. Then with d denoting a metric for X and with d_κ denoting a metric for $\mathbb{T}^\kappa$, for each $j \in \{1, 2, \ldots\}$ choose i_j so that:

$$d\bigl(f\bigl(t_{i_j}^j\bigr), x^j\bigr) < \frac{1}{j} \quad \text{and} \quad d_\kappa\bigl(f_A\bigl(x^j\bigr), \pi^\kappa\bigl(A\bigl(t_{i_j}^j\bigr)\bigr)\bigr) < \frac{1}{j}.$$

Then $\lim_j\{f(t_{i_j}^j)\}_{j=1}^\infty = x$ and

$$f_A(x) = \lim_j \bigl\{\pi^\kappa\bigl(A\bigl(t_{i_j}^j\bigr)\bigr)\bigr\}_{j=1}^\infty = \lim_j \bigl\{f_A\bigl(x^j\bigr)\bigr\}_{j=1}^\infty,$$

demonstrating that f_A is continuous. And if $A, B \in \mathcal{E}_f$ we have for $x = \lim_i\{f(t_i)\}$

$$\begin{aligned} f_{A+B}(x) &= \lim_i \bigl\{\pi^\kappa\bigl((A + B)(t_i)\bigr)\bigr\} \\ &= \lim_i \bigl\{\pi^\kappa\bigl(A(t_i)\bigr)\bigr\} + \lim_i \bigl\{\pi^\kappa\bigl(B(t_i)\bigr)\bigr\} = f_A(x) + f_B(x), \end{aligned}$$

demonstrating that ι is a homomorphism.

(2) Since $[f_A] = [f_B]$, we have that $[f_{(A-B)}] = [\text{constant map}]$, and so there is a map $g : (\overline{f(\mathbb{R}^\kappa)}, f(\mathbf{0})) \to (\mathbb{R}^\kappa, \mathbf{0})$ lifting $f_{(A-B)}$ making the following diagram commute

$$\begin{tikzcd}
 & \mathbb{R}^\kappa \arrow[dr, "\pi^\kappa"] & \\
\overline{f(\mathbb{R}^\kappa)} \arrow[ur, "g"] \arrow[rr, "f_{(A-B)}"] & & \mathbb{T}^\kappa
\end{tikzcd}$$

Then for any $t \in (A-B)(\mathcal{C})$ we have

$$f_{(A-B)} \circ f \circ (A-B)_{\mathcal{C}}^{-1}(t) = \pi^\kappa\big((A-B)(A-B)_{\mathcal{C}}^{-1}(t)\big) = \pi^\kappa(t),$$

and so we are led to the following commutative diagram:

$$\begin{tikzcd}
(A-B)(\mathcal{C}) \arrow[r, "(A-B)_{\mathcal{C}}^{-1}"] \arrow[d, "\pi^\kappa"'] & \mathcal{C} \arrow[r, "f"] & f(\mathcal{C}) \arrow[r, "g"] \arrow[dr, "f_{(A-B)}"] & \mathbb{R}^\kappa \arrow[d, "\pi^\kappa"] \\
\mathbb{T}^\kappa \arrow[rrr, "\mathrm{id}"] & & & \mathbb{T}^\kappa
\end{tikzcd}$$

Now $g \circ f \circ (A-B)_{\mathcal{C}}^{-1}$ and $\mathrm{id}_{\mathbb{R}^\kappa}$ both map $\mathbf{0}$ to $\mathbf{0}$, and so $g \circ f \circ (A-B)_{\mathcal{C}}^{-1} = \mathrm{id}_{\mathbb{R}^\kappa}$ since both provide a lift of $\mathrm{id}_{\mathbb{T}^1} \circ \pi^\kappa|_{(A-B)(\mathcal{C})}$ and such a lift is uniquely determined. And so if $\{t_i\}_{i=1}^\infty \subset \mathcal{C}$ is an f-sequence with $\lim_i\{f(t_i)\} = x \in \overline{f(\mathbb{R}^\kappa)}$, we must have

$$\begin{aligned} g(x) &= g\Big(\lim_i\big\{f \circ (A-B)_{\mathcal{C}}^{-1}\big((A-B)_{\mathcal{C}}(t_i)\big)\big\}\Big) \\ &= \lim_i\big\{g \circ f \circ (A-B)_{\mathcal{C}}^{-1}\big((A-B)_{\mathcal{C}}(t_i)\big)\big\} = \lim_i\big\{(A-B)_{\mathcal{C}}(t_i)\big\} \quad \text{in } \mathbb{R}^\kappa \end{aligned}$$

by the continuity of g.

(3) Suppose then that $\overline{f(\mathbb{R}^\kappa)}$ is compact and that $\iota(A) = [f_A] = [\text{constant map}]$ and that $v \in \mathbb{R}^\kappa - \ker A$. Then with $\mathcal{C}$ denoting the vector subspace $\mathbb{R} \cdot v \subset \mathbb{R}^\kappa$, we have that $A_{\mathcal{C}} = (A-0)_{\mathcal{C}}$ is an isomorphism onto $\mathbb{R} \cdot A(v)$. And since $\overline{f(\mathbb{R}^\kappa)}$ is compact, there is a subsequence $\{f(n_i v)\}_{i=1}^\infty$ of the sequence $\{f(nv)\}_{n=1}^\infty$ which converges to some $x \in \overline{f(\mathbb{R}^\kappa)}$. But then by the above, we must have that $\{n_i A(v)\}_{i=1}^\infty$ converges in $\mathbb{R}^\kappa$. And so if w_ℓ is any non-zero component of $A(v) = \langle w_i\rangle_{i=1}^\kappa$, we would then have that $\{n_i w_\ell\}_{i=1}^\infty$ converges in $\mathbb{R}$, which is impossible since $\{n_i\}_{i=1}^\infty$ is unbounded. Thus, we must have $\ker A = \mathbb{R}^\kappa$ and A is the zero map. That $\mathcal{E}_f$ is countable then follows from the fact that $[\overline{f(\mathbb{R}^\kappa)}; \mathbb{T}^\kappa]$ is countable.

(4) If $\kappa < \infty$ and if $\{t_i\}_{i=1}^\infty$ is an unbounded f-sequence, then $\{(A-B)(t_i)\}_{i=1}^\infty$ would be unbounded if $(A-B)$ were invertible. □

We know that when f has an image which is not compact, $\mathcal{E}_f$ may not be countable and ι may not be an embedding. For example, for $f : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{T}^1$; $\langle t_1, t_2\rangle \mapsto \langle t_1, \pi^1(t_2)\rangle$, making the identification of $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ with $\mathrm{Mat}(2, \mathbb{R})$ (the group of 2×2 matrices with real entries under component-wise addition), we have

$$\left\{\begin{pmatrix} c & 0 \\ 0 & z \end{pmatrix} \in \mathrm{Mat}(2, \mathbb{R}) \mid c \in \mathbb{R} \text{ and } z \in \mathbb{Z}\right\} \subset \mathcal{E}_f$$

and

$$\iota\left(\begin{pmatrix} c & 0 \\ 0 & z \end{pmatrix}\right) = \iota\left(\begin{pmatrix} c' & 0 \\ 0 & z \end{pmatrix}\right)$$

for any c and c' in $\mathbb{R}$. And since there are unbounded f-sequences, we know that

$$\begin{pmatrix} c & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} c' & 0 \\ 0 & z \end{pmatrix}$$

is not invertible.

Theorem 3.5. *If $f:\mathbb{R}^\kappa \to X$ has exponent group $\mathcal{E}_f = \{A_\lambda \mid \lambda \in \Lambda\}$, we have the map*

$$h_f : \overline{f(\mathbb{R}^\kappa)} \to \prod_{\lambda\in\Lambda} \mathbb{T}^\kappa, \quad x \mapsto \langle f_{A_\lambda}(x)\rangle_{\lambda\in\Lambda},$$

and $h_f \circ f$ is a homomorphism of $\mathbb{R}^\kappa$ into $\prod_{\lambda\in\Lambda}\mathbb{T}^\kappa$. And if $\overline{f(\mathbb{R}^\kappa)}$ is compact, then $h_f(\overline{f(\mathbb{R}^\kappa)})$ is a subgroup of $\prod_{\lambda\in\Lambda}\mathbb{T}^\kappa$.

Proof. Since each f_{A_λ} is continuous, we know that h_f is also continuous. When we give $\prod_{\lambda\in\Lambda}\mathbb{T}^\kappa$ the group structure of component-wise addition, it is a topological group with the product (Tychonoff) topology and

$$\begin{aligned} h_f \circ f(s+t) &= \langle \pi^\kappa \circ A_\lambda(s+t)\rangle_{\lambda\in\Lambda} = \langle \pi^\kappa \circ A_\lambda(s)\rangle_{\lambda\in\Lambda} + \langle \pi^\kappa \circ A_\lambda(t)\rangle_{\lambda\in\Lambda} \\ &= h_f \circ f(s) + h_f \circ f(t), \end{aligned}$$

and so $h_f \circ f$ is a homomorphism. When $\overline{f(\mathbb{R}^\kappa)}$ is compact, we have that $h_f(\overline{f(\mathbb{R}^\kappa)}) = \overline{h_f \circ f(\mathbb{R}^\kappa)}$ (recall that Λ is countable when $\overline{f(\mathbb{R}^\kappa)}$ is compact). We now know that $h_f \circ f(\mathbb{R}^\kappa)$ is a subgroup of $\prod_{\lambda\in\Lambda}\mathbb{T}^\kappa$, and so $\overline{h_f \circ f(\mathbb{R}^\kappa)}$ is also a subgroup. $\square$

We are now in a position to give a generalization of almost periodic maps $\mathbb{R} \to X$ to almost periodic maps $\mathbb{R}^\kappa \to X$.

Definition 3.6. A map $f:\mathbb{R}^\kappa \to X$ is almost periodic if $\mathcal{E}_f$ is countable and if the following condition is satisfied:

$$\left[\{t_i\}_{i=1}^\infty \text{ is an } f\text{-sequence}\right] \Leftrightarrow \left[\left\{\pi^\kappa\left(A(t_i)\right)\right\}_{i=1}^\infty \text{ converges for all } A \in \mathcal{E}_f\right]. \qquad (*)$$

It is known that any classically defined almost periodic function satisfies $(*)$ when $\kappa = 1$ and X is complete, see [5]. And below we shall see that any map satisfying $(*)$ is indeed almost periodic in the classical sense when $\kappa = 1$ and similar techniques may be used to show that generally the above definition coincides with Bochner's [3].

Theorem 3.7. *If f is almost periodic, then $\overline{f(\mathbb{R}^\kappa)}$ is compact, and h_f as in Theorem* 3.5 *is a homeomorphism of $\overline{f(\mathbb{R}^\kappa)}$ onto the subgroup $h_f(\overline{f(\mathbb{R}^\kappa)})$ of $\prod_{n=1}^\infty \mathbb{T}^\kappa$.*

Proof. Let $\{x^j\}_{j=1}^\infty$ be a sequence in $\overline{f(\mathbb{R}^\kappa)}$. Choose a subsequence of $\{h_f(x^j)\}_{j=1}^\infty$ which converges in $\prod_{n=1}^\infty \mathbb{T}^\kappa$ and which we label $\{h_f(y^j)\}_{j=1}^\infty$ for convenience. With

$y^j = \lim_i \{f(t_i^j)\}_{i=1}^\infty$ and with $\mathcal{E}_f = \{A_1, A_2, \ldots\}$, for each $j \in \{1, 2, \ldots\}$ choose $t_{i_j}^j$ such that

$$d\big(f\big(t_{i_j}^j\big), y^j\big) < \frac{1}{j} \quad \text{and}$$

$$\max\big\{d_{\mathbb{T}^\kappa}\big(\pi^\kappa\big(A_n\big(t_{i_j}^j\big)\big), f_{A_n}\big(y^j\big)\big) \mid n \in \{1, \ldots, j\}\big\} < \frac{1}{j}.$$

Since we have that $\{f_{A_n}(y^j)\}_{j=1}^\infty$ converges for each $n \in \{1, 2, \ldots\}$, we must then have that $\{\pi^\kappa(A(t_{i_j}^j))\}_{j=1}^\infty$ converges for all $A \in \mathcal{E}_f$, and so $\{t_{i_j}^j\}_{j=1}^\infty$ is an f-sequence by $(*)$, say $\{f(t_{i_j}^j)\}_{j=1}^\infty \to y \in X$. And so we also have $\{f(y^j)\}_{j=1}^\infty \to y$, demonstrating that $\overline{f(\mathbb{R}^\kappa)}$ is compact. Now suppose that $h_f(x) = h_f(y)$ with $x = \lim_i\{f(s_i)\}$ and $y = \lim_i\{f(t_i)\}$. We then have that for each $n \in \{1, 2, \ldots\}$

$$f_{A_n}(x) = \lim_i \big\{\pi^\kappa\big(A_n(s_i)\big)\big\}_{i=1}^\infty = \lim_i \big\{\pi^\kappa\big(A_n(t_i)\big)\big\}_{i=1}^\infty = f_{A_n}(y),$$

and so with $u_{2i-1} = s_i$ and $u_{2i} = t_i$ for $i \in \{1, 2, \ldots\}$, we have that

$$\lim_i \big\{\pi^\kappa\big(A_n(u_i)\big)\big\}_{i=1}^\infty = f_{A_n}(x)$$

for each $n \in \{1, 2, \ldots\}$. Then by $(*)$ we have that $\{f(u_i)\}_{i=1}^\infty$ converges in X, which is only possible if $x = y$. Thus, h_f is one-to-one and so is a homeomorphism onto its image. $\square$

When f is almost periodic, there is a topological isomorphism $\mathfrak{i}_f$ from $h_f(\overline{f\mathbb{R}^\kappa})$ onto some λ-solenoid $\Sigma_{\overline{M}}$ $(\lambda \leqslant \infty)$ since $h_f(\overline{f\mathbb{R}^\kappa})$ is a compact connected Abelian group; see [11, Theorem 68], [10, V.8.16] and [4,5]. Then, using the notation of [4], we have a map $\mathfrak{h} : (\mathbb{R}^\kappa, \mathbf{0}) \to (\mathbb{R}^\lambda, \mathbf{0})$ making the following diagram commute:

$$\begin{tikzcd} & \mathbb{R}^\lambda \arrow[d, "\pi_{\overline{M}}"] \\ \mathbb{R}^\kappa \arrow[ur, "\mathfrak{h}"] \arrow[r, "\mathfrak{i}_f \circ h_f \circ f"'] & \Sigma_{\overline{M}} \end{tikzcd}$$

since $\pi_{\overline{M}}$ is a fibration with unique path lifting. Then for $s, t \in \mathbb{R}^\kappa$, since $\mathfrak{i}_f \circ h_f \circ f$ and $\pi_{\overline{M}}$ are homomorphisms, we have

$$\begin{aligned} \pi_{\overline{M}}\big(\mathfrak{h}(s) + \mathfrak{h}(t)\big) &= \pi_{\overline{M}} \circ \mathfrak{h}(s) + \pi_{\overline{M}} \circ \mathfrak{h}(t) = \mathfrak{i}_f \circ h_f \circ f(s) + \mathfrak{i}_f \circ h_f \circ f(t) \\ &= \mathfrak{i}_f \circ h_f \circ f(s + t) = \pi_{\overline{M}} \circ \mathfrak{h}(s + t), \end{aligned}$$

and so $\mathfrak{h}(s) + \mathfrak{h}(t) - \mathfrak{h}(s+t) \in \ker \pi_{\overline{M}}$, implying that $\mathfrak{h}(s) + \mathfrak{h}(t) = \mathfrak{h}(s+t)$ for all $s, t \in \mathbb{R}^\kappa$, see [4, 3.4]. Hence, $\mathfrak{h}(\mathbb{R}^\kappa)$ is a subgroup of $\mathbb{R}^\lambda$. And so we may think of $\pi_{\overline{M}} \circ \mathfrak{h}(\mathbb{R}^\kappa)$ as a "linear subspace" of $\Sigma_{\overline{M}}$.

Theorem 3.8. *If $f : \mathbb{R}^\kappa \to X$ is almost periodic, then f may be extended to a continuous group action of $(\mathbb{R}^\kappa, +)$ on all of $\overline{f(\mathbb{R}^\kappa)}$:*

$$\alpha_f : \mathbb{R}^\kappa \times \overline{f(\mathbb{R}^\kappa)} \to \overline{f(\mathbb{R}^\kappa)}, \quad \textit{where } \alpha_f\big(t, f(0)\big) = f(t).$$

Proof. We have the group action

$$\phi_f : \mathbb{R}^\kappa \times \Sigma_{\overline{M}} \to \Sigma_{\overline{M}}, \quad (t,x) \mapsto \pi_{\overline{M}} \circ \mathfrak{h}(t) + x.$$

For $x \in \Sigma_{\overline{M}}$ we have $\phi_f(\mathbf{0}, x) = \pi_{\overline{M}} \circ \mathfrak{h}(\mathbf{0}) + x = e_{\overline{M}} + x = x$ and for $s, t \in \mathbb{R}^\kappa$ we have

$$\begin{aligned}\phi_f(s+t,x) &= \pi_{\overline{M}} \circ \mathfrak{h}(s+t) + x = \pi_{\overline{M}} \circ \mathfrak{h}(s) + \pi_{\overline{M}} \circ \mathfrak{h}(t) + x \\ &= \phi_f\big(s, \phi_f(t,x)\big),\end{aligned}$$

and so ϕ_f is indeed a group action. We then define

$$\alpha_f(s,x) \stackrel{\text{def}}{=} (\mathfrak{i}_f \circ h_f)^{-1}\big(\phi_f\big(s, \mathfrak{i}_f \circ h_f(x)\big)\big),$$

i.e., the action conjugate via $\mathfrak{i}_f \circ h_f$ to ϕ_f. It then follows directly that α_f is a group action, and

$$\begin{aligned}\alpha_f\big(t, f(0)\big) &= (\mathfrak{i}_f \circ h_f)^{-1}\big(\phi_f\big(t, \mathfrak{i}_f \circ h_f\big(f(0)\big)\big)\big) = (\mathfrak{i}_f \circ h_f)^{-1}\big(\pi_{\overline{M}} \circ \mathfrak{h}(t) + e_{\overline{M}}\big) \\ &= (\mathfrak{i}_f \circ h_f)^{-1}\big(\mathfrak{i}_f \circ h_f \circ f(t)\big) = f(t). \qquad \square\end{aligned}$$

And when $\kappa = 1$, α_f is a flow on $\overline{f(\mathbb{R}^\kappa)}$ and $\mathfrak{i}_f \circ h_f$ provides an equivalence of α_f and an almost periodic linear flow as described in [4], implying that f is itself almost periodic in the classical sense since it is an orbit of this flow.

Generally, it is now clear that all ϕ_f-orbits are translates of $\pi_{\overline{M}} \circ \mathfrak{h}(\mathbb{R}^\kappa)$, and so we consider the decomposition of $\Sigma_{\overline{M}}$ determined by ϕ_f as a decomposition into "linear subspaces". Thus, f determines a group action on $\overline{f(\mathbb{R}^\kappa)}$ which is equivalent to a "linear foliation" of $\Sigma_{\overline{M}}$. It then follows that if $f : \mathbb{R}^n \to M$ is a smooth almost periodic map onto a leaf of an n-dimensional foliation $\mathcal{F}$ of the manifold M that $\mathfrak{i}_f \circ h_f$ then provides an equivalence of $\mathcal{F}|_{\overline{f(\mathbb{R}^n)}}$ with the linear foliation ϕ_f since the leaves of $\mathcal{F}|_{\overline{f(\mathbb{R}^n)}}$ will coincide with the orbits of α_f due to the density of $f(\mathbb{R}^n)$ in $\overline{f(\mathbb{R}^n)}$.

Given $\boldsymbol{v} \in \mathbb{R}^\kappa$, any map $f : \mathbb{R}^\kappa \to X$ induces the map

$$\mathbb{R} \approx t \cdot \boldsymbol{v} \hookrightarrow \mathbb{R}^\kappa \xrightarrow{f} X$$

and the one-dimensional exponent group of this map $\mathbb{R} \to X$ is frequently sufficient to determine as much about $\check{H}^1(\overline{f(\mathbb{R}^\kappa)}) \approx [\overline{f(\mathbb{R}^\kappa)}; \mathbb{T}^1]$ as $\mathcal{E}_f$ reveals. But when f extends to a group action, f_A will be a fiber bundle projection, and some of the bundle structure is lost in considering only the one-dimensional maps. Thus, the higher dimensional groups have content not captured in the one-dimensional exponent group.

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